Jets, arcs, and cylinders

Eric Walker cew028@uark.edu

Jet Spaces / Arc Spaces Learning Seminar: UCSD

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Outline

Resources:

- M. Mustaţă, *Spaces of arcs in birational geometry.*
- T. de Fernex, *The space of arcs of an algebraic variety.*

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- Quick review of functors of points
- Jet spaces
- Arc spaces
- Cylinders
- The Birational Transformation Theorem
- Computing log canonical thresholds using jets and arcs

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Conventions:

- k is an algebraically closed field of characteristic 0
- $m \in \mathbf{N} \cup \{0\}$
- X is a scheme of finite type over k
- For a category $\mathcal{C}$, $Y \in \mathcal{C}$ means Y lives in the class $\operatorname{obj} \mathcal{C}$

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In our setting, we'll define schemes via their functors of points, and verify their existence via explicit construction.

Jet spaces

Jet spaces

Let $X \in \mathbf{Sch}_k^{ft}$. Define the mth jet space of X, $J^m X$ (also written X_m), to be the representing object of the functor $\mathbf{Alg}_k \to \mathbf{Set}$, $A \mapsto \mathrm{Hom}_{\mathbf{Sch}_k}(\mathrm{Spec}\, A[t]/t^{m+1}, X)$. In other words, for every $A \in \mathbf{Alg}_k$, we have a functorial bijection of sets:

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Easy to check: given any X, $J^0 X$ exists and is isomorphic to X.

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Since representing objects are unique up to isomorphism, we get $J^0X \cong X$ as claimed.

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Write π_m for $\pi_{m,0} : J^m X \to J^0 X \cong X$.

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3. If $X \in \mathbf{Sch}_k^{ft}$, then X has an affine cover $U_1 \cup \cdots \cup U_r = X$.
4. For each element of the cover, J^mU_i exists by (1). Do they glue to form a scheme? Does that scheme satisfy the functor of points that J^mX must?

□

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First see a motivating example: let $X \cong \operatorname{Spec} k[x, y]/(xy)$ and let $m = 2$. By definition,

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$$\begin{aligned} \varphi(x) &\coloneqq a_0 + a_1 t + a_2 t^2 \\ \varphi(y) &\coloneqq b_0 + b_1 t + b_2 t^2 \end{aligned}$$

subject to

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Thus the map $k[x, y]/(xy) \to A[t]/t^3$ is the same as a map $k[a_0, a_1, a_2, b_0, b_1, b_2]/(a_0 b_0, a_1 b_0 + a_0 b_1, a_2 b_0 + a_1 b_1 + a_0 b_2) \to A$.
Write $k[\underline{a}, \underline{b}]/I$ for this k-algebra.

Therefore,

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This process gives a general algorithm for computing J^mX for $X \in \mathbf{AffSch}_k^{ft}$. Specifying an $A[t]/t^{m+1}$-point of X is a map $\varphi : k[x_1,\ldots,x_n]/(f_1,\ldots,f_s) \to A[t]/t^{m+1}$. Consider the images $\varphi(x_i)$ as $(m+1)$ choices of elements of A and subject to the relations $f_j(\varphi(x_1),\ldots,\varphi(x_n)) = 0$. Consequently J^mX can be defined as an affine subscheme of $\mathbf{A}^{n(m+1)}$ given by the vanishing of a set of polynomials determined by f_js.

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DERIVATIVES!

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Theorem. If $X \in \mathbf{AffSch}_k^{ft}$, i.e.,

$$X \cong \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_s),$$

then $J^m X$ exists, and moreover,

$$J^m X \cong \operatorname{Spec} {}^{k[x_i, x_i{}', x_i{}'', \ldots, x_i{}^{(m)} \mid 1 \leq i \leq n]}\!\big/_{(f_j, f_j{}', f_j{}'', \ldots, f_j{}^{(m)} \mid 1 \leq j \leq s)},$$

where we understand $f_j{}^{(\ell)}$ to mean formal implicit differentiation.

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Let $A \in \mathbf{Alg}_k$ and consider the natural homomorphism induced by π_m, $\iota_A : \operatorname{Spec} A \to \operatorname{Spec} A[t]/t^{m+1}$. An m-jet in $J^m X$, a map $f : \operatorname{Spec} A[t]/t^{m+1} \to X$, factors through V if and only if $f \circ \iota_A$ factors through V.

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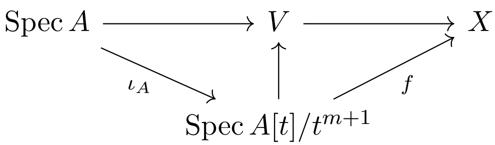

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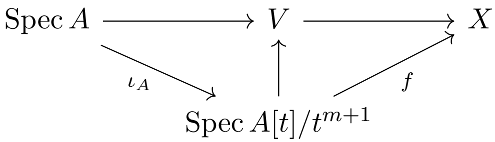

Therefore, ${\pi_m}^{-1} V$ is the set of jets in $J^m V \subseteq J^m X$, i.e., the maps $\operatorname{Spec} A[t]/t^{m+1} \to V$.

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Does the scheme we've just glued satisfy the functor of points definition that $J^m X$ must? Yes, an easy exercise for the reader.

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- If X is a nonsingular variety of dimension n, then $J^m X$ is a nonsingular variety of dimension $n(m+1)$.
- The maps $\pi_{m,p} : J^m X \to J^p X$, $m > p$, are affine morphisms of k-schemes.

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$$\cdots \to J^m X \to J^{m-1} X \to \cdots \to J^1 X \to J^0 X \cong X.$$

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By abstract nonsense, the projective limit of this diagram exists in $\mathbf{Sch}_k$.
Define the arc space of X, $J^\infty X$ (also written X_∞ and sometimes $\mathcal{L}(X)$), to be the projective limit

$$J^\infty X \coloneqq \varprojlim J^m X.$$

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If $X \in \mathbf{Sch}_k$, then any $\operatorname{Spec} k[t]/t^{m+1} \to X$ and $\operatorname{Spec} k[\![t]\!] \to X$ must factor through any affine open neighborhood of the image of the closed point. Consequently, the elements of $J^\infty X(k)$ correspond to arcs in X; i.e., we have a bijection

$$\mathrm{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} k, J^\infty X) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} k[\![t]\!], X).$$

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- **Theorem** [Kolchin]**.** If X is a variety, then $J^\infty X$ is irreducible. (X nonsingular is easy, X singular requires resolution of singularities ($\operatorname{char} k = 0$))

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Let $C = {\psi_m}^{-1}(S)$ be a cylinder. We define

$$\operatorname{codim}(C) \coloneqq \operatorname{codim}(S, J^m X) = (m+1)n - \dim(S)$$

(independent of m).

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If X is singular, bullets (1) and (4) fail, while (3) is an open problem.

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Let $Z \subseteq X$ be a proper closed subscheme. Define a function $\mathrm{ord}_Z : J^\infty X \to \mathbf{N} \cup \{0, \infty\}$ given by, if $\gamma : \operatorname{Spec} k[\![t]\!] \to X \in J^\infty X$, then the inverse image of the ideal defining Z is an ideal in $k[\![t]\!]$ generated by $t^{\mathrm{ord}_Z(\gamma)}$.

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The contact locus of order m with Z is defined to be the set $\mathrm{Cont}^m(Z) \coloneqq {\mathrm{ord}_Z}^{-1}(m)$. Similarly, $\mathrm{Cont}^{\geq m}(Z) \coloneqq {\mathrm{ord}_Z}^{-1}(\geq m)$.

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$$\mathrm{Cont}^{\geq m}(Z) = {\psi_{m-1}}^{-1}(J^{m-1}Z),$$

so $\mathrm{Cont}^{\geq m}(Z)$ is a closed cylinder. Also $\mathrm{Cont}^m(Z)$ is a locally closed cylinder.

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The Birational Transformation Theorem [Kontsevich] describes the behavior of contact loci defined by a particular effective divisor $K_{X/Y} \subseteq X$ for a fixed map $f : X \to Y$. We will state it, then use it to calculate log canonical thresholds using jets and arcs.

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Setup: let $f : X \to Y$ be a proper birational morphism. Let $\dim X = \dim Y = n$. Give X and Y local coordinates at $P \in X$ and $f(P) \in Y$; call them $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. Define the relative canonical divisor $K_{X/Y}$ to be the unique effective divisor obtained by local equation at $P \in X$ the determinant of the Jacobian

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_1} \\ \frac{\partial f_1}{\partial x_2} & & & \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

where $f_i \in k[\![x_1, \ldots, x_n]\!]$ is given by $f^*(y_i) = f_i(x_1, \ldots, x_n)$.

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Setup (cont.): Define a cylinder $C^{(e)} \coloneqq \mathrm{Cont}^e(K_{X/Y})$ for $e \in \mathbf{N}$. Write $\psi_m^X : J^\infty X \to J^m X$ and $\psi_m^Y : J^\infty Y \to J^m Y$. Write $\pi_{m,p}^X : J^m X \to J^p X$ and $\pi_{m,p}^Y : J^m Y \to J^p Y$.

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Theorem [Kontsevich]**.** Given the prior setup, let $m \geq 2e$.

1. Let $\gamma_m, \gamma'_m \in J^m X$. If $\gamma_m \in \psi_m^X(C^{(e)})$ and $J^m f(\gamma_m) = J^m f(\gamma'_m)$, then

$$\pi_{m,m-e}^X(\gamma_m) = \pi_{m,m-e}^X(\gamma'_m).$$

2. The induced map

$$\psi_m^X(C^{(e)}) \to J^m f(\psi_m^X(C^{(e)}))$$

is piecewise trivial with fiber $\mathbf{A}^e$.

Computing log canonical thresholds using jets and arcs

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Recall: let X be a nonsingular variety and $Y \subseteq X$ a proper closed subscheme. Let $f : X' \to X$ be a log resolution of (X, Y); i.e., f is proper and birational, X' is nonsingular, and $f^{-1}(Y) + K_{X'/X}$ has simple normal crossings. We have seen that the log canonical threshold can be defined as

$$\operatorname{lct}(X, Y) := \min_i \frac{k_i + 1}{a_i},$$

where

$$f^{-1}(Y) = \sum_{i=1}^{s} a_i E_i \text{ and } K_{X'/X} = \sum_{i=1}^{s} k_i E_i.$$

Computing log canonical thresholds using jets and arcs

Theorem [Ein-Lazarsfeld-Mustaţă]**.** Let $f : X' \to X$ be a log resolution of (X, Y) and as before write $f^{-1}(Y) = \sum a_i E_i$ and $K_{X'/X} = \sum k_i E_i$. WLOG, f is an isomorphism over $X \setminus Y$, so $f^{-1}(Y)$ is effective. For all $m \in \mathbf{N}$,

$$\operatorname{codim}(\operatorname{Cont}^m(Y)) = \min_{\nu} \sum_{i=1}^{s} (k_i + 1)\nu_i,$$

where $\nu = (\nu_i) \in \mathbf{N}^s$ such that

$$\sum_{i=1}^{s} a_i \nu_i = m \text{ and } \bigcap_{\nu_i \geq 1} E_i \neq \emptyset.$$

Computing log canonical thresholds using jets and arcs

Proof outline.

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Computing log canonical thresholds using jets and arcs

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4. Put the pieces together to complete the theorem.

□

Computing log canonical thresholds using jets and arcs

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Computing log canonical thresholds using jets and arcs

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The decomposition is

$$\begin{aligned} f^{-1}(\mathrm{Cont}^m(Y)) &= \mathrm{Cont}^m(f^{-1}(Y)) \\ &= \mathrm{Cont}^m\left(\sum_{i=1}^{s} a_i E_i\right) \\ &= \coprod_{\nu}\left(\bigcap_{i=1}^{s} \mathrm{Cont}^{\nu_i}(E_i)\right), \end{aligned}$$

where $\nu = (\nu_i)$ and

$$\sum_{i=1}^{s} a_i \nu_i = m.$$

We'll write $\mathrm{Cont}^{\nu}(E)$ for $\bigcap \mathrm{Cont}^{\nu_i}(E_i)$.

Computing log canonical thresholds using jets and arcs

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Our decomposition is $f^{-1}(\mathrm{Cont}^m(Y)) = \coprod \mathrm{Cont}^\nu(E)$. Since $\sum E_i$ has simple normal crossings, to compute $\mathrm{codim}(\mathrm{Cont}^\nu(E))$, we may take an étale morphism to $\mathbf{A}^n$ so that E_i is a hyperplane in an affine space. Using this we see that $\mathrm{Cont}^\nu(E) \neq \emptyset$ if and only if

$$\bigcap_{\nu_i \geq 1} E_i \neq \emptyset,$$

and in this case

$$\mathrm{codim}(\mathrm{Cont}^\nu(E)) = \sum_{i=1}^{s} \nu_i.$$

Computing log canonical thresholds using jets and arcs

3. After that use Kontsevich's Birational Transformation Theorem to compute the contact loci of the relative canonical divisor $K_{X'/X}$.

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Note that $\mathrm{Cont}^{\nu}(E) \subseteq \mathrm{Cont}^{e}(K_{X'/X})$ where $e \coloneqq \sum k_i v_i$. Let $p \gg 0$. By [Kontsevich] (1), $\psi_p^X(\mathrm{Cont}^{\nu}(E))$ is a union of fibers of $J^p f$. By [Kontsevich] (2),

$$\mathrm{codim}(J^{\infty} f(\mathrm{Cont}^{\nu}(E))) = \sum_{i=1}^{s} (k_i + 1)\nu_i.$$

Computing log canonical thresholds using jets and arcs

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Computing log canonical thresholds using jets and arcs

4 Put the pieces together to complete the theorem.

Since $f^{-1}(\mathrm{Cont}^m(Y)) = \coprod \mathrm{Cont}^\nu(E)$, we also have a decomposition $\mathrm{Cont}^m(Y) = \coprod J^\infty f(\mathrm{Cont}^\nu(E))$ (**Proposition:** $J^\infty f$ is a bijection over $\mathrm{Cont}^m(Y)$). Therefore,

$$\begin{aligned}\operatorname{codim}(\mathrm{Cont}^m(Y)) &= \min_\nu \operatorname{codim}(J^\infty f(\mathrm{Cont}^\nu(E))) \\ &= \min_\nu \sum_{i=1}^{s} (k_i+1)\nu_i,\end{aligned}$$

as desired.

Computing log canonical thresholds using jets and arcs

Corollary. If X is a nonsingular variety and $Y \subseteq X$ is a proper closed subscheme, then

$$\mathrm{lct}(X,Y) \coloneqq \min_i \frac{k_i+1}{a_i} = \dim(X) - \max_m \frac{\dim(J^m Y)}{m+1}.$$

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Proof.

[ELM] implies that $\operatorname{codim}(\mathrm{Cont}^{\geq m}(Y)) = \min_\nu \sum (k_i + 1)\nu_i$, where $\nu = (\nu_i)$ satisfies $m \leq \sum a_i \nu_i$.

Computing log canonical thresholds using jets and arcs

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For all i, $\operatorname{lct}(X, Y) a_i \leq k_i + 1$ by definition.

Computing log canonical thresholds using jets and arcs

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Proof (cont.).

Hence

$$\begin{aligned} m \operatorname{lct}(X,Y) &\leq \operatorname{codim}(\operatorname{Cont}^{\geq m}(Y)) \\ &= \operatorname{codim}(J^{m-1}Y, J^{m-1}X) \\ &= m\dim(X) - \dim(J^{m-1}Y). \end{aligned}$$

Computing log canonical thresholds using jets and arcs

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Proof (cont.).

Let ℓ be the index that realizes $\operatorname{lct}(X, Y) = (k_\ell + 1)/a_\ell$. Let ν be $\nu_\ell \geq 1$ and $\nu_i = 0$ for $i \neq \ell$, then

$$\operatorname{codim}(\operatorname{Cont}^{\geq a_\ell \nu_\ell}(Y)) \leq a_\ell \nu_\ell \operatorname{lct}(X, Y).$$

Computing log canonical thresholds using jets and arcs

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Thus $\dim(J^{m-1}Y) \geq m(\dim(X) - \operatorname{lct}(X, Y))$ if a_ℓ divides m. Rearrange and the result is shown. □

Computing log canonical thresholds using jets and arcs

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A quick jaunt to Macaulay2 confirms

$$\begin{aligned}
\dim(J^0 V(xy)) &= \dim(V(xy)) = 1, \\
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so

$$\mathrm{lct}(\mathbf{A}^2, V(xy)) = 2 - \max\left\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \dots\right\} = 2 - 1 = 1.$$

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We calculate

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so

$$\mathrm{lct}(\mathbf{A}^2, V(x^2 - y^3)) = 2 - \max\left\{1, 1, 1 \ldots, \frac{7}{6}, \ldots\right\} = 2 - \frac{7}{6} = \frac{5}{6}.$$

Computing log canonical thresholds using jets and arcs

Feel free to double check my computation of $\dim(J^5V(x^2-y^3))$ in M2:

```
i1 :  R=QQ[x0,x1,x2,x3,x4,x5,y0,y1,y2,y3,y4,y5]
i2 :  I=ideal((x0)^2-(y0)^3,
      2*x0*x1-3*(y0)^2*y1,
      2*x0*x2+2*(x1)^2-3*(y0)^2*y2-6*y0*(y1)^2,
      2*x0*x3+6*x1*x2-3*(y0)^2*y3-6*(y1)^3-18*y0*y1*y2,
      2*x0*x4+6*(x2)^2+8*x3*x1-3*y4*(y0)^2-18*y0*(y2)^2-24*y3*y0*y1-36*(y1)^2*y2,
      2*x0*x5+10*x4*x1+20*x3*x2-3*y5*(y0)^2-60*y3*y0*y2-60*y3*(y1)^2-30*y1*y0*y4-90*y1*(y2)^2)
i3 :  dim(I)
```